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# Action of classical operators on $so_q(3) \subset u_q(3)$ basis states

P P Raychev†‡, R P Roussev‡, N Lo Iudice† and P A Terziev‡

† Dipartimento di Scienze Fisiche, Università di Napoli 'Federico II', Mostra d'Oltremare, Pad. 19, I-80125 Napoli, Italy

‡ Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tzarigrad Road, BG-1784 Sofia, Bulgaria

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**Abstract.** A method for expanding the $q$-deformed $su_q(3)$ states into a classical $so(3)$ basis has been developed. This expansion is used to compute the matrix elements of classical irreducible tensor operators between $q$-deformed states. Examples of scalar products between deformed and classical basis states and some of their properties are discussed.

## 1. Introduction

In a series of papers [1–8] a prescription for constructing basis states and $q$-deformed irreducible tensor operators for the chain $u_q(3) \supset so_q(3)$ was given in the case of the most symmetric representation $[\lambda, 0, 0]$ of $u_q(3)$. In our previous paper [9] a simplified realization of the basis states was used to compute the reduced matrix elements of a special second-rank tensor operator ($q$-deformed quadrupole operator). This scheme has a physical valence only if observables and physical states can be described by $q$-deformed operators and wavefunctions.

One may take a less extreme view and assume that the evolution of a physical system is determined by a Hamiltonian of a classical form. In this conventional scheme, all observables must be described by classical (undeformed) operators. Even in this context, it is still legitimate to use $q$-deformed wavefunctions as long as they are viewed as (effective) highly correlated eigenfunctions of the (undeformed) Hamiltonian.

In the present paper we take this latter view to describe the electromagnetic properties of the ground band of a deformed system. We assume that the states of this band are $q$-deformed $u_q(3)$ states in the symmetric representation $[\lambda, 0, 0]$, while the electromagnetic multipole operators are just classical irreducible tensor operators. We consider only quadrupole transitions, which are of the utmost importance for the study of rotational spectra in deformed nuclei. In order to achieve such a program first we shall recall some known facts about the reduction $u_q(3) \supset so_q(3)$ in the framework of the symmetric representations of $u_q(3)$ (section 2) and then expand the resulting states in terms of the classical $so(3)$ basis (section 3). A short conclusion follows. Finally, some technical aspects are illustrated in appendices A and B.

## 2. $u_q(3) \supset so_q(3)$ basis states

It has been shown by Van der Jeugt [1, 2] that in the space of the *most symmetric* irreducible representation $[\lambda, 0, 0]$ of the $q$-deformed $u_q(3)$ algebra one can define three operators

  

$L_+$, $L_0$, $L_-$, which separate a $q$-deformed $so_q(3)$ subalgebra of $u_q(3)$. Then the $so_q(3)$ generators, corresponding to the embedding $u_q(3) \supset so_q(3)$, have the form [1]

$$\begin{aligned} L_0 &= N_+ - N_- \\ L_+ &= q^{N_- - \frac{1}{2}N_0}\sqrt{q^{N_+} + q^{-N_+}}\, b_+^\dagger b_0 + b_0^\dagger b_- q^{N_+ - \frac{1}{2}N_0}\sqrt{q^{N_-} + q^{-N_-}} \\ L_- &= b_0^\dagger b_+ q^{N_- - \frac{1}{2}N_0}\sqrt{q^{N_+} + q^{-N_+}} + q^{N_+ - \frac{1}{2}N_0}\sqrt{q^{N_-} + q^{-N_-}}\, b_-^\dagger b_0 \end{aligned} \tag{2.1}$$

and satisfy the relations

$$[L_0, L_\pm] = \pm L_\pm \qquad [L_+, L_-] = [2L_0]. \tag{2.2}$$

Herein, $[x] = (q^x - q^{-x})/(q - q^{-1})$ and the parameter $q$ is taken to be generic. In the notation used in this paper the $so_q(3)$ generators are expressed in terms of three independent $q$-deformed boson operators $b_i$ and $b_i^\dagger$, satisfying [10–12]

$$[N_i, b_i^\dagger] = b_i^\dagger \qquad [N_i, b_i] = -b_i \qquad b_i b_i^\dagger - q^{\pm 1} b_i^\dagger b_i = q^{\mp N_i} \qquad i = +, 0, - \tag{2.3}$$

where $N_i$ are the corresponding number operators. Although (2.1) is not a subalgebra of $u_q(3)$ considered as a Hopf algebra, in our context only the $q$-deformed relations for the algebras are used.

In [9] it was shown that the operators (2.1) can be expressed in a simplified form if one introduces the following 'modified' $q$-deformed operators

$$B_0 = q^{-\frac{1}{2}N_0} b_0 \qquad B_i = q^{N_i + \frac{1}{2}} b_i \sqrt{\frac{[2N_i]}{[N_i]}} \qquad i = +, - \tag{2.4}$$

and the corresponding conjugate operators

$$B_0^\dagger = b_0^\dagger q^{-\frac{1}{2}N_0} \qquad B_i^\dagger = \sqrt{\frac{[2N_i]}{[N_i]}}\, b_i^\dagger q^{N_i + \frac{1}{2}} \qquad i = +, -. \tag{2.5}$$

Note that (2.5) are Hermitian conjugates of (2.4) only if the parameter $q$ is real. The operators (2.4) and (2.5) satisfy the standard commutation relations with the number operators

$$[N_i, B_i^\dagger] = B_i^\dagger \qquad [N_i, B_i] = -B_i \qquad i = +, 0, -. \tag{2.6}$$

Likewise, in the Fock space spanned on the normalized eigenstates of the number operators $N_+$, $N_0$, $N_-$, they satisfy the relations

$$[B_0, B_0^\dagger] = q^{-2N_0} \qquad [B_i, B_i^\dagger] = [2]q^{4N_i + 1} \qquad i = +, - \tag{2.7}$$

and all other commutation relations are zero.

In terms of the 'modified' operators (2.4) and (2.5) the $so_q(3)(L_0, L_\pm)$ and $sp_{q^2}(2, R)(S_0, (1/[2])S_\pm)$ generators, corresponding to the embedding $so_q(3) \oplus sp_{q^2}(2, R) \subset sp_q(6, R)$, take the simplified form [3]

$$\begin{aligned} L_0 &= N_+ - N_- & S_0 &= 2^{-1}(N + 3/2) \\ L_+ &= q^{-L_0 + \frac{1}{2}} B_+^\dagger B_0 + q^{L_0 - \frac{1}{2}} B_0^\dagger B_- & S_+ &= (B_0^\dagger)^2 q^{2S_0} - B_+^\dagger B_-^\dagger q^{-2S_0} \\ L_- &= q^{-L_0 + \frac{1}{2}} B_+^\dagger B_+ + q^{L_0 + \frac{1}{2}} B_-^\dagger B_0 & S_- &= q^{2S_0}(B_0)^2 - q^{2S_0} B_+ B_-. \end{aligned} \tag{2.8}$$

Now we shall recall the expression for the common eigenvectors of $N$, $L_0$ and the Casimir operator of $so_q(3)$, $C_2^{(q)} = L_- L_+ + [L_0][L_0 + 1]$, in terms of the 'modified' operators $B_m^\dagger$. Then we shall express $B_m^\dagger$ in terms of the 'standard' (non-deformed) boson operators $a_m^\dagger$ ($[a_m, a_n^\dagger] = \delta_{nm}$). In this way we can find a connection between the $q$-deformed and 'standard' angular momentum states.

Following [1, 2] and using the results of [9], the normalized $so_q(3)$ basis states, which are characterized by the $q$-deformed angular momentum $L$, its projection $M$ and belong to the symmetric $[\lambda, 0, 0]$ irrep of $u_q(3)$, can be written in the form

$$\left| \begin{array}{l} \lambda \\ L, M \end{array} \right\rangle_q = \frac{q^{-\frac{1}{2}L^2}}{N_{\lambda L}^{(q)}} \sqrt{\frac{[L+M]!}{[2L]![L-M]!}} (S_+)^{\frac{1}{2}(\lambda-L)} (L_-)^{L-M} \frac{(B_+^\dagger)^L}{\sqrt{[2L]!!}} |0\rangle \qquad (2.9)$$

where $L = \lambda, \lambda-2, \ldots, 0$ or $1$, $[x]! = [x][x-1]\ldots[1]$ and $[x]!! = [x][x-2]\ldots[2]$ or $[1]$. The normalization constant $N_{\lambda L}^{(q)}$ in (2.9) equals

$$N_{\lambda L}^{(q)} = \sqrt{\frac{[\lambda-L]!![\lambda+L+1]!!}{[2L+1]!!}} \qquad (2.10)$$

and the $so_q(3)$ scalar operator $S_+$ has the form

$$S_+ = \underbrace{(B_0^\dagger)^2 q^{2S_0}}_{\xi} - \underbrace{B_+^\dagger B_-^\dagger q^{-2S_0}}_{\eta} \qquad \eta\xi = q^{-4}\xi\eta. \qquad (2.11)$$

In order to represent the term $(S_+)^{\frac{1}{2}(\lambda-L)}$ in powers of $B_i^\dagger$ we use the $q$-*binomial theorem* [13], according to which, if the elements $A$ and $B$ satisfy the condition $BA = qAB$ then

$$(A-B)^k = \sum_{t=0}^{k} (-1)^t q^{\frac{1}{2}t(k-t)} \begin{bmatrix} k \\ t \end{bmatrix}_{q^{1/2}} A^{k-t} B^t \qquad \text{where } \begin{bmatrix} k \\ t \end{bmatrix}_q = \frac{[k]_q!}{[t]_q![k-t]_q!}. \qquad (2.12)$$

Therefore, for the power of $S_+$ we obtain

$$(S_+)^k = q^{k(k+\frac{1}{2})}[2k]!! \sum_{t=0}^{k} \frac{(-1)^t q^{-(2k+1)t}}{[2t]!![2k-2t]!!} (B_+^\dagger)^t (B_0^\dagger)^{2(k-t)} (B_-^\dagger)^t q^{(k-2t)N}. \qquad (2.13)$$

Using the identity

$$\frac{(L_-)^{L-M}}{[L-M]!} \frac{(B_+^\dagger)^L}{[2L]!!} |0\rangle = q^{\frac{1}{2}(L^2-M^2)} \sum_{x=\max(0,M)}^{\lfloor (L+M)/2 \rfloor} \frac{(B_+^\dagger)^x}{[2x]!!} \frac{(B_0^\dagger)^{L+M-2x}}{[L+M-2x]!} \frac{(B_-^\dagger)^{x-M}}{[2x-2M]!!} |0\rangle \qquad (2.14)$$

and the expansion (2.13) of $(S_+)^k$ in powers of $B_i^\dagger$, the states (2.9) can be written in the form [1, 3, 9]

$$\begin{aligned} \left| \begin{array}{l} \lambda \\ L, M \end{array} \right\rangle_q &= q^{\frac{1}{4}(\lambda-L)(\lambda+L+1)-\frac{1}{2}M^2} \sqrt{\frac{[\lambda-L]!![L+M]![L-M]![2L+1]}{[\lambda+L+1]!!}} \\ &\quad \times \sum_{t=0}^{(\lambda-L)/2} \sum_{x=\max(0,M)}^{\lfloor (L+M)/2 \rfloor} \frac{(-1)^t q^{-(\lambda+L+1)t}}{[2t]!![\lambda-L-2t]!!} \frac{(B_+^\dagger)^{x+t}}{[2x]!!} \\ &\quad \times \frac{(B_0^\dagger)^{\lambda+M-2x-2t}}{[L+M-2x]!} \frac{(B_-^\dagger)^{x+t-M}}{[2x-2M]!!} |0\rangle. \end{aligned} \qquad (2.15)$$

Furthermore, we shall represent the 'modified' $q$-operators $B_i^\dagger$ in terms of the 'standard' boson operators using the well known construction of $q$-deformed bosons $b_i^\dagger$, $b_i$ in terms of the classical ones $a_i^\dagger$, $a_i$ [10–12]

$$b_i^\dagger = \sqrt{\frac{[N_i]}{N_i}} a_i^\dagger \qquad b_i = a_i \sqrt{\frac{[N_i]}{N_i}} \qquad i = +, 0, -. \qquad (2.16)$$

In this way the operators $B_i^\dagger$, $B_i$ can be written in the form

$$\begin{aligned} B_0 &= q^{-\frac{1}{2}N_0} a_0 \sqrt{\frac{[N_0]}{N_0}} \qquad & B_i &= q^{N_i+\frac{1}{2}} a_i \sqrt{\frac{[2N_i]}{N_i}} \\ B_0^\dagger &= \sqrt{\frac{[N_0]}{N_0}} a_0^\dagger q^{-\frac{1}{2}N_0} \qquad & B_i^\dagger &= \sqrt{\frac{[2N_1]}{N_i}} a_i^\dagger q^{N_i+\frac{1}{2}} \end{aligned} \qquad i = +, -. \tag{2.17}$$

As a next step we can replace $B_i^\dagger$ in (2.15) with their expressions from (2.17). Then, using the following identity, which holds in the Fock space,

$$\frac{(B_+^\dagger)^x}{\sqrt{[2x]!!}} \frac{(B_0^\dagger)^y}{\sqrt{[y]!}} \frac{(B_-^\dagger)^z}{\sqrt{[2z]!!}} |0\rangle = q^{\frac{1}{2}(x^2+z^2)-\frac{1}{4}y(y-1)} \frac{(a_+^\dagger)^x}{\sqrt{x!}} \frac{(a_0^\dagger)^y}{\sqrt{y!}} \frac{(a_-^\dagger)^z}{\sqrt{z!}} |0\rangle \tag{2.18}$$

we obtain [3, 5]

$$\begin{aligned} \left| \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q &= q^{-\frac{1}{4}\{(2\lambda+M)(M-1)+L(L+1)\}} N_{\lambda LM}^{(q)} \sum_{t=0}^{(\lambda-L)/2} \sum_{x=\max(0,M)}^{\lfloor (L+M)/2 \rfloor} \\ &\quad \times \frac{(-1)^t q^{(\lambda-\frac{1}{2})x-(L+\frac{3}{2})t}}{[2t]!![\lambda-L-2t]!!} \frac{\sqrt{[2x+2t]!![\lambda+M-2x-2t]![2x+2t-2M]!!}}{[2x]!![L+M-2x]![2x-2M]!!} \\ &\quad \times \frac{(a_+^\dagger)^{x+t}}{\sqrt{(x+t)!}} \frac{(a_0^\dagger)^{\lambda+M-2x-2t}}{\sqrt{(\lambda+M-2x-2t)!}} \frac{(a_-^\dagger)^{x+t-M}}{\sqrt{(x+t-M)!}} |0\rangle \end{aligned} \tag{2.19}$$

where the normalization factor $N_{\lambda LM}^{(q)}$ is

$$N_{\lambda LM}^{(q)} = \sqrt{\frac{[\lambda-L]!![L+M]![L-M]![2L+1]}{[\lambda+L+1]!!}}. \tag{2.20}$$

The expression (2.19) will be used for the calculation of the matrix elements of ‘standard’ (or classical) tensor operators (which are built up from standard boson operators) between the states with given $q$-deformed angular momentum.

Consider now the classical quadrupole operator $Q_m^{(c)}$. Its zero component is of the form

$$\begin{aligned} Q_0^{(c)} &= \sqrt{6}[a^\dagger \otimes \tilde{a}]_{20} = a_+^\dagger \tilde{a}_- + 2a_0^\dagger \tilde{a}_0 + a_-^\dagger \tilde{a}_+ \\ &= -a_+^\dagger a_+ + 2a_0^\dagger a_0 - a_-^\dagger a_- = 3N_0 - N. \end{aligned} \tag{2.21}$$

Here $a_m^\dagger$, $a_m (m = +, 0, -)$ are classical boson operators and $\tilde{a}_m = (-1)^m a_{-m}$.

The explicit expression for the matrix elements of $Q_0^{(c)}$ between the $q$-deformed states (2.19) then becomes

$${}_q\left\langle \begin{matrix} \lambda \\ J, M' \end{matrix} \right| Q_0^{(c)} \left| \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q = \delta_{M,M'} \; {}_q\left\langle \begin{matrix} \lambda \\ J, M \end{matrix} \right| Q_0^{(c)} \left| \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q \tag{2.22}$$

and

$$\begin{aligned} {}_q\left\langle \begin{matrix} \lambda \\ J, M \end{matrix} \right| Q_0^{(c)} \left| \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q &= (2\lambda+3M)\delta_{L,J} - 6q^{-\frac{1}{2}(2\lambda+M)(M-1)-\frac{1}{4}\{L(L+1)+J(J+1)\}} \\ &\quad \times N_{\lambda LM}^{(q)} N_{\lambda JM}^{(q)} \sum_{t=0}^{(\lambda-L)/2} \sum_{r=0}^{(\lambda-J)/2} \frac{(-1)^{t+r} q^{-(\lambda+L+1)t-(\lambda+J+1)r}}{[2t]!![\lambda-L-2t]!![2r]!![\lambda-J-2r]!!} \\ &\quad \times \sum_x \frac{[2x]!![\lambda+M-2x]![2x-2M]!!}{[2x-2t]!![L+M-2x+2t]![2x-2t-2M]!!} \\ &\quad \times \frac{x q^{(2\lambda-1)x}}{[2x-2r]!![J+M-2x+2r]![2x-2r-2M]!!} \end{aligned} \tag{2.23}$$

where $\max\{t, r, M+t, M+r\} \leqslant x \leqslant \min\{\lfloor (L+M)/2 \rfloor + t, \lfloor (J+M)/2 \rfloor + r\}$. Obviously, the recipe, described above, can be applied for the calculation of matrix elements between the $q$-deformed angular momentum states of any classical operator. The disadvantage of this method is that one must perform special calculations for any specific operator. This can be avoided by the use of the transformation matrix between the 'standard' and $q$-deformed states—a problem considered in the next section.

## 3. Transformation between $so_q(3)$ and $so(3)$ basis states

For fixed $\lambda$, one can expand the $q$-deformed basis states (2.19) in terms of classical ones (when $q \to 1$), since both sets form orthonormal bases in the eigenspace $\mathcal{H}_\lambda$ of the number operator $N = N_+ + N_0 + N_-$, corresponding to the eigenvalue $\lambda$. In particular, $\dim \mathcal{H}_\lambda = \frac{1}{2}(\lambda+1)(\lambda+2)$ and the operator $L_0$ has the same form $L_0 = N_+ - N_-$ in both classical and $q$-deformed cases. In this way

$$\left| \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q = \sum_{J, M_J} \left| \begin{matrix} \lambda \\ J, M_J \end{matrix} \right\rangle_{\mathrm{c}} {}_{\mathrm{c}}\left\langle \begin{matrix} \lambda \\ J, M_J \end{matrix} \right| \left. \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q = \sum_{J} \left| \begin{matrix} \lambda \\ J, M \end{matrix} \right\rangle_{\mathrm{c}} {}_{\mathrm{c}}\left\langle \begin{matrix} \lambda \\ J, M \end{matrix} \right| \left. \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q \tag{3.1}$$

where $J = \lambda, \lambda-2, \ldots, |M|$ or $|M|+1$ and the subscript 'c' denotes the classical (non-deformed) basis states. Taking the scalar product between $q$-deformed (2.19) and classical basis states, for the transformation matrix in (3.1) we obtain

$$\begin{aligned} {}_{\mathrm{c}}\left\langle \begin{matrix} \lambda \\ J, M' \end{matrix} \right| \left. \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q &= \delta_{M,M'} q^{-\frac{1}{4}\{(2\lambda+M)(M-1)+L(L+1)\}} N^{(\mathrm{c})}_{\lambda J M} N^{(q)}_{\lambda L M} \\ &\times \sum_{t=0}^{(\lambda-L)/2} \sum_{r=0}^{(\lambda-J)/2} \frac{(-1)^{t+r} q^{-(\lambda+L+1)t}}{[2t]!![\lambda-L-2t]!!(2r)!!(\lambda-J-2r)!!} \\ &\times \sum_x \frac{q^{(\lambda-\frac{1}{2})x} \sqrt{[2x]!![\lambda+M-2x]![2x-2M]!!}}{[2x-2t]!![L+M-2x+2t]![2x-2t-2M]!!} \\ &\times \frac{\sqrt{(2x)!!(\lambda+M-2x)!(2x-2M)!!}}{(2x-2r)!!(J+M-2x+2r)!(2x-2r-2M)!!} \end{aligned} \tag{3.2}$$

where $\max\{t, r, M+t, M+r\} \leqslant x \leqslant \min\{\lfloor (L+M)/2 \rfloor + t, \lfloor (J+M)/2 \rfloor + r\}$. It should be noted that the scalar product (3.2) is real and has the properties

$${}_{\mathrm{c}}\left\langle \begin{matrix} \lambda \\ J, M' \end{matrix} \right| \left. \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q = \delta_{M,M'} \, {}_{\mathrm{c}}\left\langle \begin{matrix} \lambda \\ J, M \end{matrix} \right| \left. \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q = {}_{q}\left\langle \begin{matrix} \lambda \\ L, M \end{matrix} \right| \left. \begin{matrix} \lambda \\ J, M' \end{matrix} \right\rangle_{\mathrm{c}} \tag{3.3}$$

which we have used in (3.1). Although looking very involved, the formula (3.2) is quite handy for computational purposes. Indeed, only a few terms are contained in the sum. The same equation shows that the transformation matrix is diagonal in the third projections of $J$ and $L$, and is non-vanishing even for high values of the difference $\Delta(J) = |L - J|$. However, the values are peaked around the classical value $\Delta(J) = 0$ if the parameter of deformation $q$ tends to unity.

Consider now a classical tensor operator $A^{(\mathrm{c})}_{jm}$ of rank $j$ according to the algebra $so(3)$, which conserves the number of particles. Using the Wigner–Eckart theorem in the classical case we can express the matrix elements of $A^{(c)}_{jm}$ between the classical $so(3)$ basis states in the form

$$ {}_{\rm c}\left\langle \begin{matrix} \lambda \\ J, M' \end{matrix} \right| A^{(c)}_{jm} \left| \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_{\rm c} = (-1)^{2j} \frac{C^{JM'}_{LM,jm}}{\sqrt{2J+1}} \langle \lambda J || A^{(c)}_{j} || \lambda L \rangle \tag{3.4} $$

where $C^{JM'}_{LM,jm}$ are the classical Clebsch–Gordan coefficients of the $so(3)$ algebra, and we assume that the classical reduced matrix elements of $A^{(c)}_{j}$ in (3.4) are known. In this way, from (3.1) it follows that

$$ \begin{aligned} A^{(c)}_{jm} \left| \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q &= \sum_R A^{(c)}_{jm} \left| \begin{matrix} \lambda \\ R, M \end{matrix} \right\rangle_{\rm c} {}_{\rm c}\left\langle \begin{matrix} \lambda \\ R, M \end{matrix} \middle| \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q \\ &= \sum_R \sum_{P,M_P} \left| \begin{matrix} \lambda \\ P, M_P \end{matrix} \right\rangle_{\rm c} {}_{\rm c}\left\langle \begin{matrix} \lambda \\ P, M_P \end{matrix} \right| A^{(c)}_{jm} \left| \begin{matrix} \lambda \\ R, M \end{matrix} \right\rangle_{\rm c} {}_{\rm c}\left\langle \begin{matrix} \lambda \\ R, M \end{matrix} \middle| \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q \end{aligned} \tag{3.5} $$

and using (3.4) we have the expansion

$$ \begin{aligned} {}_q\left\langle \begin{matrix} \lambda \\ J, M' \end{matrix} \right| A^{(c)}_{jm} \left| \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q &= \sum_R \sum_P (-1)^{2j} \frac{C^{PM'}_{RM,jm}}{\sqrt{2P+1}} \; {}_q\left\langle \begin{matrix} \lambda \\ J, M' \end{matrix} \middle| \begin{matrix} \lambda \\ P, M' \end{matrix} \right\rangle_{\rm c} \\ &\quad \times \langle \lambda P \| A^{(c)}_{j} || \lambda R \rangle \; {}_{\rm c}\left\langle \begin{matrix} \lambda \\ R, M \end{matrix} \middle| \begin{matrix} \lambda \\ L, M \end{matrix} \right\rangle_q . \end{aligned} \tag{3.6} $$

In particular, equation (3.6) shows that the matrix elements of the classical tensor operator $A^{(c)}_{jm}$ between $q$-deformed $so_q(3)$ states are non-vanishing if $M' = M + m$, as in the classical basis. It is also worth remarking that the summation expression (3.6) gives a universal way for (numerical) computation of matrix elements of different types of classical operators between deformed basis states (2.19). For instance, it is straightforward to show that the numerical values of the quadrupole operator calculated by (2.23) of the previous section and the method used in this section (involving transformation matrices) coincide. However, the method descibed in this section is universal and can be applied to any particular classical operators.

## 4. Conclusion

In order to be able to compute matrix elements of *classical* operators between $q$-deformed $u_q(3)$ states we first constructed a simplified realization of the $so_q(3)$ subalgebra of $u_q(3)$, restricted to symmetric representations, and then expanded the corresponding $so_q(3)$ states in terms of the classical $so(3)$ basis. This expansion should be of considerable help in the study of the structure of $q$-deformed $u_q(3)$ states. The results obtained could possibly be of practical interest for clarifying the role of $q$-deformation in the study of nuclear rotational spectra [14–16] and their electromagnetic properties. It is known, for instance, that in the classical $su(3)$ scheme the nuclear rotational band terminates at a critical value of the angular momentum, which is lower than the one observed experimentally. A numerical analysis should enable us to ascertain if by the use of $q$-deformed $su(3)$ states the critical value is pushed upward. Another aspect worth of attention is represented by the deviations induced by the $q$-deformation on the angular momentum dependence of the E2 strength with respect to the corresponding laws predicted in the geometrical or classical $su(3)$ models. Finally, $q$-deformation will enable us to study the E2 forbidden transitions among the members of the band, a problem without immediate solution in conventional approaches. Investigations along these lines are under way.

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## Appendix A. Expressions for angular momentum states in oscillator basis

The oscillator basis can be expressed as

$$
\begin{aligned}
|x, y, z\rangle &= \frac{(a_+^\dagger)^x}{\sqrt{x!}} \frac{(a_0^\dagger)^y}{\sqrt{y!}} \frac{(a_-^\dagger)^z}{\sqrt{z!}} |0\rangle = \frac{(b_+^\dagger)^x}{\sqrt{[x]!}} \frac{(b_0^\dagger)^y}{\sqrt{[y]!}} \frac{(b_-^\dagger)^z}{\sqrt{[z]!}} |0\rangle \\
&= q^{-\frac{1}{2}(x^2+z^2)+\frac{1}{4}y(y-1)} \frac{(B_+^\dagger)^x}{\sqrt{[2x]!!}} \frac{(B_0^\dagger)^y}{\sqrt{[y]!}} \frac{(B_-^\dagger)^z}{\sqrt{[2z]!}} |0\rangle
\end{aligned}
\tag{A.1}
$$

where $a_m^\dagger$, $b_m^\dagger$ and $B_m^\dagger$ are 'standard', $q$-deformed and 'modified' boson creation operators, the last two determined by (2.16) and (2.17). Equation (A.1) reflects the fact that the oscillator basis built up in terms of 'standard', $q$-deformed and 'modified' boson operators coincide [10–12].

According to (2.19) the $q$-deformed state with angular momentum $L$ and maximal projection $M = L$ can be written down in the form

$$
\begin{aligned}
\left| \begin{matrix} L+2k \\ L, L \end{matrix} \right\rangle_q &= q^{k(L+1)} \sqrt{\frac{[2k]!![2L+1]!!}{[2k+2L+1]!![2L]!!}} \sum_{t=0}^{k} \frac{(-1)^t q^{-\frac{1}{2}(2L+3)t}}{[2k-2t]!!} \\
&\quad \times \sqrt{\frac{[2L+2t]!![2k-2t]!}{[2t]!!}} |L+t, 2k-2t, t\rangle.
\end{aligned}
\tag{A.2}
$$

Equation (A.2) gives the expression of the highest weight $q$-deformed state in terms of oscillator states $|x, y, z\rangle$. On the other hand, the 'classical' angular momentum states can also be expanded in the oscillator basis as follows:

$$
\begin{aligned}
\left| \begin{matrix} J+2k' \\ J, M \end{matrix} \right\rangle_{\mathrm{c}} &= \sqrt{\frac{(2k')!!(J+M)!(J-M)!(2J+1)}{(2k'+2J+1)!!}} \sum_{s=0}^{k'} \frac{(-1)^s}{(2s)!!(2k'-2s)!!} \\
&\quad \times \sum_{p=\max(0,M)}^{\lfloor \frac{1}{2}(J+M) \rfloor} \frac{\sqrt{(2p+2s)!!(2k'+J+M-2p-2s)!(2p+2s-2M)!!}}{(2p)!!(J+M-2p)!(2p-2M)!!} \\
&\quad \times |p+s, 2k'+J+M-2p-2s, p+s-M\rangle.
\end{aligned}
\tag{A.3}
$$

Taking into account the orthonormality of the oscillator basis one can verify that

$$
{}_{\mathrm{c}}\left\langle \begin{matrix} \lambda' \\ J, M \end{matrix} \right| \left. \begin{matrix} \lambda \\ L, L \end{matrix} \right\rangle_q = \delta_{\lambda,\lambda'} \delta_{L,M} \; {}_{\mathrm{c}}\left\langle \begin{matrix} \lambda \\ J, L \end{matrix} \right| \left. \begin{matrix} \lambda \\ L, L \end{matrix} \right\rangle_q .
\tag{A.4}
$$

In this way for the scalar product of (A.2) and (A.3) (i.e. for the transformation bracket in (A.4)) we obtain

$$
{}_{\mathrm{c}}\left\langle \begin{matrix} L+2k \\ J, L \end{matrix} \right| \left. \begin{matrix} L+2k \\ L, L \end{matrix} \right\rangle_q = q^{k(L+1)} \sqrt{\frac{[2k]!![2L+1]!!}{[2k+2L+1]!![2L]!!}}
$$

$$\times\sqrt{\frac{(2k')!!(J+L)!(J-L)!(2J+1)}{(2k'+2J+1)!!}}$$
$$\times\sum_{t=0}^{k}\sum_{s=\max(0,t')}^{\min(k',t)}\frac{(-1)^{t+s}q^{-\frac{1}{2}(2L+3)t}}{[2k-2t]!!(2s)!!(2k'-2s)!!}\sqrt{\frac{[2L+2t]!![2k-2t]!}{[2t]!!}}$$
$$\times\frac{\sqrt{(2L+2t)!!(2k-2t)!(2t)!!}}{(2L+2t-2s)!!(J-L-2t+2s)!(2t-2s)!!} \tag{A.5}$$

where

$$k'=(L+2k-J)/2=k-(J-L)/2 \qquad J=L,L+2,\ldots,L+2k$$
$$t'=t-(J-L)/2=t-k+k' \qquad \lambda'=J+2k'=L+2k=\lambda.$$

## Appendix B. Examples for some simple transformation brackets

Here we illustrate two simple examples of expansions of highest weight ($M = L$) $q$-deformed angular momentum states (A.2) in terms of classical ones. Although these expansions can directly be computed from equation (A.5), we shall give and further use their explicit expressions in terms of oscillator states.

First let us consider the expansion of the $q$-deformed state with $\lambda = L+2$ in terms of oscillator basis states

$$\left|\begin{matrix}L+2\\L,L\end{matrix}\right\rangle_q=\frac{1}{\sqrt{[2L+3]}}\{q^{L+1}|L,2,0\rangle-q^{-\frac{1}{2}}\sqrt{[2L+2]}|L+1,0,1\rangle\}. \tag{B.1}$$

On the other hand, for the 'classical' (non-deformed) states we have

$$\left|\begin{matrix}L+2\\L,L\end{matrix}\right\rangle_c=\frac{1}{\sqrt{2L+3}}\{|L,2,0\rangle-\sqrt{2L+2}|L+1,0,1\rangle\}$$
$$\left|\begin{matrix}L+2\\L+2,L\end{matrix}\right\rangle_c=\frac{1}{\sqrt{2L+3}}\{\sqrt{2L+2}|L,2,0\rangle+|L+1,0,1\rangle\}. \tag{B.2}$$

Therefore the $q$-deformed state (B.1) can be represented as a linear combination of 'classical' states as follows:

$$\left|\begin{matrix}L+2\\L,L\end{matrix}\right\rangle_q=\alpha_0^{(1,L)}\left|\begin{matrix}L+2\\L,L\end{matrix}\right\rangle_c+\alpha_1^{(1,L)}\left|\begin{matrix}L+2\\L+2,L\end{matrix}\right\rangle_c. \tag{B.3}$$

From (B.1) and (B.2) we obtain a system of equations, with solutions

$$\alpha_0^{(1,L)}={}_c\left\langle\begin{matrix}L+2\\L,L\end{matrix}\right|\left.\begin{matrix}L+2\\L,L\end{matrix}\right\rangle_q=C\{q^{L+1}+\sqrt{(2L+2)[2L+2]}q^{-\frac{1}{2}}\} \tag{B.4}$$

$$\alpha_1^{(1,L)}={}_c\left\langle\begin{matrix}L+2\\L+2,L\end{matrix}\right|\left.\begin{matrix}L+2\\L,L\end{matrix}\right\rangle_q=C\{\sqrt{2L+2}q^{L+1}-\sqrt{[2L+2]}q^{-\frac{1}{2}}\} \tag{B.5}$$

where

$$C=\frac{1}{\sqrt{(2L+3)[2L+3]}}$$

and one can verify that the condition $(\alpha_0^{(1,L)})^2+(\alpha_1^{(1,L)})^2=1$ holds.

In the same way, in the case $\lambda = L+4$, we have the expansion

$$\left|\begin{matrix}L+4\\L,L\end{matrix}\right\rangle_q=\alpha_0^{(2,L)}\left|\begin{matrix}L+4\\L,L\end{matrix}\right\rangle_c+\alpha_1^{(2,L)}\left|\begin{matrix}L+4\\L+2,L\end{matrix}\right\rangle_c+\alpha_2^{(2,L)}\left|\begin{matrix}L+4\\L+4,L\end{matrix}\right\rangle_c \tag{B.6}$$

where

$$\alpha_0^{(2,L)} = C_0\left\{q^{2L+2}\sqrt{3[3]} + 2q^{L+\frac{1}{2}}\sqrt{\frac{[4][2L+2](L+1)}{[2]}}\right.$$
$$\left.+2q^{-1}\sqrt{[2L+4][2L+2](L+2)(L+1)}\right\} \tag{B.7}$$

$$\alpha_1^{(2,L)} = C_1\left\{2q^{2L+2}\sqrt{3[3](L+1)} + (2L+1)q^{L+\frac{1}{2}}\sqrt{\frac{[4][2L+2]}{[2]}}\right.$$
$$\left.-2q^{-1}\sqrt{[2L+4][2L+2](L+2)}\right\} \tag{B.8}$$

$$\alpha_2^{(2,L)} = C_2\left\{2q^{2L+2}\sqrt{[3](L+1)(L+2)} - 2q^{L+\frac{1}{2}}\sqrt{\frac{3[4][2L+2](L+2)}{[2]}}\right.$$
$$\left.+q^{-1}\sqrt{3[2L+4][2L+2]}\right\} \tag{B.9}$$

and

$$C_0 = \frac{B}{\sqrt{(2L+3)(2L+5)}} \qquad C_1 = \frac{B}{\sqrt{(2L+3)(2L+7)}}$$
$$C_2 = \frac{B}{\sqrt{(2L+5)(2L+7)}} \qquad B = \frac{1}{\sqrt{[2L+3][2L+5]}}.$$

Again one can verify the condition $(\alpha_0^{(2,L)})^2 + (\alpha_1^{(2,L)})^2 + (\alpha_2^{(2,L)})^2 = 1$.

At the end of this appendix we shall give an alternative form of the expansion (B.3), using the zero component of the classical quadrupole operator $Q^{(c)}$. Returning to the expression (2.21) for $Q_0^{(c)}$, it follows that the action of this operator on the classical highest weight $so(3)$ states is

$$Q_0^{(c)}\left|\begin{matrix}\lambda\\ L,L\end{matrix}\right\rangle_c = \boldsymbol{a}\left|\begin{matrix}\lambda\\ L,L\end{matrix}\right\rangle_c + \boldsymbol{b}\left|\begin{matrix}\lambda\\ L+2,L\end{matrix}\right\rangle_c \tag{B.10}$$

where

$$\boldsymbol{a} = -\frac{(2\lambda+3)L}{2L+3} \qquad \boldsymbol{b} = \frac{6}{2L+3}\sqrt{\frac{(\lambda-L)(\lambda+L+3)(L+1)}{2L+5}}. \tag{B.11}$$

Combining (B.3) and (B.10) in the case $\lambda = L+2$, one finds

$$\left|\begin{matrix}L+2\\ L,L\end{matrix}\right\rangle_q = \{\beta_0^{(1,L)} + \beta_1^{(1,L)}Q_0^{(c)}\}\left|\begin{matrix}L+2\\ L,L\end{matrix}\right\rangle_c \tag{B.12}$$

where

$$\beta_0^{(1,L)} = \frac{1}{6}\sqrt{\frac{2L+3}{[2L+3]}}\left\{(L+2)q^{L+1} - (L-4)\sqrt{\frac{[2L+2]}{2L+2}}q^{-1/2}\right\} \tag{B.13}$$

$$\beta_1^{(1,L)} = \frac{1}{6}\sqrt{\frac{2L+3}{[2L+3]}}\left\{q^{L+1} - \sqrt{\frac{[2L+2]}{2L+2}}q^{-1/2}\right\}. \tag{B.14}$$

In particular, from (B.13) and (B.14) it follows that when $q$ tends to unity the coefficient (B.14) vanishes, which recovers the classical state.

It is worth mentioning, that formula (B.12) reveals a remarkable property of the $q$-deformed states, namely that the particular $q$-deformed state with $L = \lambda - 2$ can be obtained through the action of the classical $Q_0^{(\mathrm{c})}$ on the classical states. In this sense, the action of $Q_0^{(\mathrm{c})}$ generates the deformation of the classical state. Consideration of the most general case with $L = \lambda - 2k$ is in progress.

## References

[1] Van der Jeugt J 1992 *J. Phys. A: Math. Gen.* **25** L213
[2] Van der Jeugt J 1993 *Group Theoretical Methods in Physics (Salamanca 1992), Anales de Física, Monografías)* vol 1, ed M A del Olmo, M Santander and J Mateos Guilarte (Madrid: Real Sociedad Española de Física) p 131
[3] Van der Jeugt J 1993 *J. Math. Phys.* **34** 1799
[4] Van der Jeugt J 1993 Deformed $u_q(3)$ algebra in an $so_q(3)$ basis *Preprint* TWI-93-30, University of Gent
[5] Van der Jeugt J 1994 *Can. J. Phys.* **72** 519
[6] Quesne C 1993 *Phys. Lett.* **304B** 81
[7] Sciarrino A 1993 Deformation of Lie algebras in a non-Chevalley basis and 'embedding' of $q$-algebras *Preprint* DSF-T-18/93, University of Napoli
[8] Sciarrino A 1994 'Deformed $U(gl(3))$' from $so_q(3)$ *Symmetries in Science VII: Spectrum Generating Algebras and Dynamics in Physics* ed B Gruber (New York: Plenum)
[9] Raychev R, Roussev R, Terziev P, Bonatsos D and Lo Iudice N 1996 *J. Phys. A: Math. Gen.* **29** 6939
[10] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[11] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[12] Kulish P and Damaskinsky E 1990 *J. Phys. A: Math. Gen.* **23** L415
[13] Nomura M 1989 *J. Math. Phys.* **30** 2397
Nomura M 1989 *J. Phys. Soc. Japan* **59** 439
Nomura M 1989 *J. Phys. Soc. Japan* **59** 2345
Nomura M 1990 *J. Phys. Soc. Japan* **60** 789
[14] Raychev P 1995 Quantum groups: application to nuclear and molecular spectroscopy *Adv. Quant. Chem.* **26** 239
[15] Raychev P, Roussev R and Smirnov Yu 1990 *J. Phys. G: Nucl. Part. Phys.* **16** L137
[16] Bonatsos D, Drenska S, Raychev P, Roussev R and Smirnov Yu 1991 *J. Phys. G: Nucl. Part. Phys.* **17** L67